

# SYMMETRY AND LINEAR STABILITY IN SERRIN'S OVERDETERMINED PROBLEM VIA THE STABILITY OF THE PARALLEL SURFACE PROBLEM

GIULIO CIRAOLO, ROLANDO MAGNANINI, AND VINCENZO VESPRI

ABSTRACT. We consider the solution of the problem

$$-\Delta u = f(u) \quad \text{and} \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma,$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with boundary  $\Gamma$  of class  $C^{2,\tau}$ ,  $0 < \tau < 1$ , and  $f$  is a locally Lipschitz continuous non-linearity. Serrin's celebrated symmetry theorem states that, if the normal derivative  $u_\nu$  is constant on  $\Gamma$ , then  $\Omega$  must be a ball.

In [CMS2], it has been conjectured that Serrin's theorem may be obtained *by stability* in the following way: first, for a solution  $u$  prove the estimate

$$r_e - r_i \leq C_\delta [u]_{\Gamma^\delta}$$

for some constant  $C_\delta$  depending on  $\delta > 0$ , where  $r_e$  and  $r_i$  are the radii of a spherical annulus containing  $\Gamma$ ,  $\Gamma^\delta$  is a surface parallel to  $\Gamma$  at distance  $\delta$  and sufficiently close to  $\Gamma$ , and  $[u]_{\Gamma^\delta}$  is the Lipschitz semi-norm of  $u$  on  $\Gamma^\delta$ ; secondly, if in addition  $u_\nu$  is constant on  $\Gamma$ , show that

$$[u]_{\Gamma^\delta} = o(C_\delta) \quad \text{as} \quad \delta \rightarrow 0^+.$$

In this paper, we prove that this strategy is successful.

As a by-product of this method, for  $C^{2,\tau}$ -regular domains, we also obtain a linear stability estimate for Serrin's symmetry result. Our result is optimal and greatly improves the similar logarithmic-type estimate of [ABR] and the Hölder estimate of [CMV] that was restricted to convex domains.

## 1. INTRODUCTION

In this paper we establish a connection between two overdetermined boundary value problems, Serrin's symmetry problem and what we call the *parallel surface problem*. As a consequence, we obtain optimal stability for the former, thus significantly improving previous results ([ABR], [CMV]).

Serrin's symmetry problem concerns solutions of elliptic partial differential equations subject to both Dirichlet and Neumann boundary conditions. Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with sufficiently smooth boundary  $\Gamma$  (say  $C^{2,\tau}$ ,  $0 < \tau < 1$ ). As shown in one of his seminal papers, [Se], if the following problem

$$(1.1) \quad \Delta u + f(u) = 0 \quad \text{and} \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma,$$

$$(1.2) \quad u_\nu = \alpha \quad \text{on} \quad \Gamma,$$

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admits a solution for a given positive constant  $\mathfrak{a}$ , then  $\Omega$  must be a ball. Here,  $f : [0, +\infty) \rightarrow \mathbb{R}$  is a locally Lipschitz continuous function and  $\nu$  is the *inward* unit vector field to  $\Gamma$ . Generalizations of these result are innumerable and we just mention [BCN], [BNV], [GL], [GNN], [Re].

The parallel surface problem concerns solutions of (1.1) with a level surface *parallel* to  $\Gamma$ , that is to say the solution  $u$  of (1.1) is required to be constant at a fixed distance from  $\Gamma$ :

$$(1.3) \quad u = \mathfrak{b} \quad \text{on} \quad \Gamma^\delta.$$

Here,

$$(1.4) \quad \Gamma^\delta = \{x \in \Omega : d_\Omega(x) = \delta\} \quad \text{with} \quad d_\Omega(x) = \min_{y \notin \Omega} |x - y|, \quad x \in \mathbb{R}^N,$$

is a surface parallel to  $\Gamma$ , and  $\mathfrak{b}$  and  $\delta$  are positive (sufficiently small) constants. Under sufficient conditions on  $\Gamma^\delta$ , also in this case, if a solution of (1.1) and (1.3) exists, then  $\Omega$  must be a ball (see [MS2, MS3, Sh, CMS1, GGS]). In the sequel, we will occasionally refer to problem (1.1), (1.3) as the *parallel surface problem*.

A condition like (1.3) was considered in [MS2, MS3] in connection with *time-invariant* level surfaces of a solution  $v$  of the non-linear equation

$$v_t = \Delta \phi(v) \quad \text{in} \quad \Omega \times (0, \infty)$$

subject to the initial and boundary conditions

$$v = 1 \quad \text{on} \quad \Omega \times \{0\} \quad \text{and} \quad v = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty);$$

here,  $\phi$  is a  $C^2$ -smooth non-linearity with  $\phi(0) = 0$  and  $\phi'$  bounded from below and above by two positive constants (hence, we are dealing with a non-degenerate fast-diffusion equation). A (spatial) level surface  $\Sigma \subset \Omega$  of  $v$  is time-invariant if  $v$  is constant on  $\Sigma$  for each fixed time  $t > 0$ .

It is proved in [MS2] that for  $x \in \overline{\Omega}$

$$4t \int_0^{v(x,t)} \frac{\phi'(\eta)}{1-\eta} d\eta \longrightarrow d_\Omega(x)^2 \quad \text{as} \quad t \rightarrow 0^+,$$

uniformly on compact subsets of  $\Omega$ . Hence, if  $\Sigma$  is time-invariant, then it has to be parallel to  $\Gamma$  at some distance  $\delta$ . Also, it is not difficult to show that the function

$$u(x) = \int_0^\infty \phi(v(x,t)) dt, \quad x \in \overline{\Omega},$$

satisfies (1.1), with  $f(u) = 1$  and, being  $\Sigma$  time-invariant,  $u$  satisfies (1.3), with  $\Gamma^\delta = \Sigma$ , for some positive constant  $\mathfrak{b}$ . As a consequence,  $\Omega$  is a ball and, in this situation, the time-invariant surface  $\Sigma$  turns out to be a sphere.

Condition (1.3) can also be re-interpreted to give a connection to transnormal and isoparametric functions and surfaces. We recall that, in differential geometry, a function  $u$  is *transnormal* in  $\Omega$  if it is a solution of the equation

$$(1.5) \quad |Du|^2 = g(u) \quad \text{in} \quad \Omega,$$

for some suitably smooth function  $g : \mathbb{R} \rightarrow (0, \infty)$ ; the level surfaces of  $u$  are called *transnormal surfaces*. A transnormal function that also satisfies the first equation in (1.1) is called an *isoparametric function* and its level surfaces are *isoparametric surfaces*. Isoparametric surfaces in the euclidean space can only be (portions of) spheres, spherical cylinders or hyperplanes ([L-C], [Sg]);

a list of essential references about transnormal and isoparametric functions and their properties in other spaces includes [Bo], [Ca], [Mi], [Wa].

Now, notice that the (viscosity) solution  $u$  of (1.5) such that  $u = 0$  on  $\Gamma$  takes the form  $u(x) = h(\text{dist}(x, \partial\Omega))$ , where  $h$  is defined by

$$\int_0^{h(t)} \frac{ds}{\sqrt{g(s)}} = t, \quad t \geq 0.$$

It is then clear that a solution of (1.1) satisfies (1.3) if and only if  $u = \mathfrak{b}$  on  $\{x \in \Omega : w(x) = h(\delta)\}$ . Thus, we can claim that *if a solution of (1.1) has a level surface that is also a level surface of a transnormal function, then  $\Omega$  must be a ball.*

Both problems (1.1), (1.2) and (1.1), (1.3) have at least one feature in common: the proof of symmetry relies on the *method of moving planes*, a refinement, designed by J. Serrin, of a previous idea of V.I. Aleksandrov's [Al]. The evident similarity between the two problems arouses a natural question: *to obtain the symmetry of  $\Omega$ , is condition (1.2) weaker or stronger than (1.3)?*

As noticed in [CMS2] and [CM], condition (1.3) seems to be weaker than (1.2), in the sense clarified hereafter. As (1.3) does not imply (1.2), the latter can be seen as the limit of a sequence of conditions of type (1.3) with  $\mathfrak{b} = \mathfrak{b}_n$  and  $\delta = \delta_n$  and  $\mathfrak{b}_n$  and  $\delta_n$  vanishing as  $n \rightarrow \infty$ . As (1.2) does not imply (1.3) either, nonetheless the oscillation on a surface parallel to  $\Gamma$  of a solution of (1.1), (1.2) becomes smaller than usual, the closer the surface is to  $\Gamma$ . A way to quantitatively express this fact is to consider the Lipschitz seminorm

$$[u]_{\Gamma^\delta} = \sup_{\substack{x, y \in \Gamma^\delta, \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|},$$

that controls the oscillation of  $u$  on  $\Gamma^\delta$ : it is not difficult to show by a Taylor-expansion argument (see the proof of Theorem 5.2) that, if  $u \in C^{2,\tau}(\overline{\Omega})$ ,  $0 < \tau < 1$ , satisfies (1.1), (1.2), then

$$(1.6) \quad [u]_{\Gamma^\delta} = O(\delta^{1+\tau}) \quad \text{as } \delta \rightarrow 0.$$

This remark suggests the possibility that Serrin's symmetry result may be obtained *by stability* by the following strategy: (i) for the solution  $u$  of (1.1) prove that, for some constant  $C_\delta$  depending on  $\delta$ , an estimate of type

$$(1.7) \quad r_e - r_i \leq C_\delta [u]_{\Gamma^\delta}$$

holds for any sufficiently small  $\delta > 0$ , where  $r_e$  and  $r_i$  are the radii of a spherical annulus centered at some point  $\mathfrak{o}$ ,  $\{x \in \mathbb{R}^N : r_i < |x - \mathfrak{o}| < r_e\}$ , containing  $\Gamma$  — this means that  $\Omega$  is nearly a ball, if  $u$  does not oscillate too much on  $\Gamma^\delta$ ; (ii) if in addition  $u$  satisfies (1.2), that is  $u_\nu$  is constant on  $\Gamma$ , show that

$$[u]_{\Gamma^\delta} = o(C_\delta) \quad \text{as } \delta \rightarrow 0^+.$$

The spherical symmetry of  $\Gamma$  then will follow by choosing  $\delta$  arbitrarily small.

In [CMS2], the first two authors of this paper proved an estimate of type (1.7). Unfortunately, that estimate is not sufficient for our aims, since the computed constant  $C_\delta$  blows up exponentially as  $\delta \rightarrow 0^+$ . Nevertheless, in

the case examined in [CM], we showed that our strategy is successful for the very special class of ellipses.

In this paper we shall extend the efficacy of our strategy to the class of  $C^{2,\tau}$ -smooth domains. The crucial step in this direction is Theorem 4.2, where we considerably improve inequality (1.7) by showing that it holds with a constant  $C_\delta$  that is a  $O(\delta^{-1})$  as  $\delta \rightarrow 0^+$ . Thus, (1.6) will imply that  $r_i = r_e$ , that is  $\Omega$  is a ball (see Theorem 5.2). By a little more effort, in Theorem 5.3 we will prove that

$$r_e - r_i \leq C [u_\nu]_\Gamma,$$

for some positive constant  $C$ . This inequality enhances to optimality two previous results, both also based on the method of moving planes. In fact, it improves the logarithmic stability obtained in [ABR] for  $C^{2,\tau}$ -regular domains and extends the linear stability obtained for convex domains in [CMV]. We notice that, in our inequality the seminorm  $[u_\nu]_\Gamma$  replaces the deviation in the  $C^1(\Gamma)$ -norm of the function  $u_\nu$  from a given constant, considered in [ABR]. Moreover, the inequality also improves [BNST][Theorem 1.2], where a Hölder-type estimate was obtained, for the case in which  $f(u) = -N$ , by means of integral identities.

The outline of the proof of Theorem 4.2 will be recalled in Section 2: it is the same as that of [CMS2, Theorem 4.1], that relies on ideas introduced in [ABR], and the use of *Harnack's* and *Carleson's (or the boundary Harnack's) inequalities*. In Section 3 — the heart of this paper — by the careful use of refined versions of those inequalities (see [Ba], [BCN]), we prove the necessary lemmas that in Section 4 allow us to obtain our optimal version of (1.7). Finally, in Section 5, we present our new linear stability estimate for the radial symmetry in Serrin's problem; it implies symmetry for (1.1), (1.2): thus, the new strategy is successful.

## 2. A PATH TO STABILITY:

### THE QUANTITATIVE METHOD OF MOVING PLANES

In this Section we introduce some notation and we review the quantitative study of the method of moving planes as carried out in [ABR] and [CMS2] (see also [CV]).

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $\Gamma$  be its boundary; we shall denote the diameter of  $\Omega$  by  $\mathfrak{d}_\Omega$ . The distance  $d_\Omega$  defined in (1.4) is always Lipschitz continuous on  $\overline{\Omega}$  and of class  $C^{2,\tau}$  in a neighborhood of  $\Gamma$ , if this is of class  $C^{2,\tau}$ ,  $0 < \tau < 1$ . In fact, under this assumption on  $\Gamma$ , for every  $x \in \Gamma$  there are balls  $B \subset \Omega$  such that  $x \in \partial B$ ; denote by  $r_x$  the supremum of the radii of such balls and set

$$\mathfrak{r}_\Omega = \min_{x \in \partial\Omega} r_x.$$

We then denote by  $\mathfrak{R}_\Omega$  the number obtained by this procedure, where instead the interior ball  $B$  is replaced by an exterior one.

For  $\delta > 0$ , let  $\Omega^\delta$  be the *parallel set* (to  $\Gamma$ ), i.e.

$$(2.1) \quad \Omega^\delta = \{x \in \Omega : d_\Omega(x) > \delta\}.$$

We know that if  $0 \leq \delta < \mathfrak{r}_\Omega$ , then each level surface  $\Gamma^\delta$  of  $d_\Omega$ , as defined by (1.4), is of class  $C^{2,\tau}$  and will be referred to as a *parallel surface* (to  $\Gamma$ ).

The following notations are useful to carry out the method of moving planes and its quantitative version; for  $\omega \in \mathbb{S}^{N-1}$  and  $\mu \in \mathbb{R}$ , we set:

$$(2.2) \quad \begin{aligned} \pi_\mu &= \{x \in \mathbb{R}^N : x \cdot \omega = \mu\} && \text{a hyperplane orthogonal to } \omega, \\ \mathcal{H}_\mu &= \{x \in \mathbb{R}^N : x \cdot \omega > \mu\} && \text{the half-space *on the right* of } \pi_\mu, \\ A_\mu &= A \cap \mathcal{H}_\mu && \text{the right-hand cap of a set } A, \\ x' &= x - 2(x \cdot \omega - \mu)\omega && \text{the reflection of } x \text{ in } \pi_\mu, \\ (A_\mu)' &= \{x \in \mathbb{R}^N : x' \in A_\mu\} && \text{the reflected cap in } \pi_\mu. \end{aligned}$$

In the sequel, we will generally use the simplified notation  $A' = (A_\mu)'$  every time in which the dependence on  $\mu$  is not important.

Set  $\Lambda = \sup\{x \cdot \omega : x \in \Omega\}$ ; if  $\mu < \Lambda$  is close to  $\Lambda$ , the reflected cap  $(\Omega_\mu)'$  is contained in  $\Omega$  (see [Fr]), and hence we can define the number

$$(2.3) \quad \lambda = \inf\{\mu : (\Omega_\mu)' \subset \Omega \text{ for all } \tilde{\mu} \in (\mu, \Lambda)\}.$$

Thus, at least one of the following two cases occurs ([Se],[Fr]):

- (S1)  $\Omega' = (\Omega_\lambda)'$  is internally tangent to  $\partial\Omega$  at some point  $p' \in \partial\Omega' \setminus \pi_\lambda$ , which is the reflection in  $\pi_\lambda$  of a point  $p \in \partial\Omega_\lambda \setminus \pi_\lambda$ ;
- (S2)  $\pi_\lambda$  is orthogonal to  $\partial\Omega$  at some point  $q \in \partial\Omega \cap \pi_\lambda$ .

In the sequel,  $\pi_\lambda$  and  $\Omega_\lambda$  will be referred to as the *critical hyperplane* and the *critical cap* (in the direction  $\omega$ ), respectively. Corresponding to the points  $p$  and  $q$ , we will also consider the points  $p^\delta = p + \delta \nu(p)$  and  $q^\delta = q + \delta \nu(q)$  for  $0 < \delta < \mathfrak{r}_\Omega$ ; notice that  $\Gamma^\delta = \{p^\delta : p \in \Gamma\}$ .

Let  $\mathfrak{r} \in (0, \mathfrak{r}_\Omega)$ . From now on  $G$  will denote the parallel set

$$G = \{x \in \Omega : d_\Omega(x) > \mathfrak{r}\}.$$

In Section 4, we shall choose  $\mathfrak{r}$  appropriately. Also, to simplify notations, by  $P$  and  $Q$  we shall denote  $p^\mathfrak{r}$  and  $q^\mathfrak{r}$ , respectively — two points on  $\partial G$  that will be frequently used.

Now, the function  $w$  defined by

$$(2.4) \quad w(x) = u(x') - u(x), \quad x \in \Omega_\lambda,$$

satisfies

$$\Delta w + c(x)w = 0 \quad \text{in } \Omega_\lambda,$$

where for  $x \in \Omega_\lambda$

$$c(x) = \begin{cases} \frac{f(u(x')) - f(u(x))}{u(x') - u(x)} & \text{if } u(x') \neq u(x), \\ 0 & \text{if } u(x') = u(x). \end{cases}$$

Notice that  $c(x)$  is bounded by the Lipschitz constant  $\mathfrak{L}$  of  $f$  in the interval  $[0, \frac{\max u}{\Omega}]$ .

By an argument introduced in [Se, Theorem 2] and improved in [BNV] (see also [Fr]), we can assume that  $w \geq 0$  in  $\Omega_\lambda$  and hence, by the strong maximum principle applied to the inequality  $\Delta w - c^-(x)w \leq 0$  with  $c^-(x) = \max[-c(x), 0]$ , we can suppose that

$$w > 0 \quad \text{in } \Omega_\lambda.$$

One ingredient in our estimates of Section 3 will be *Harnack's inequality*: thanks to this result, for fixed  $a \in (0, 1)$ ,  $w$  satisfies the inequality

$$(2.5) \quad \sup_{B_{ar}} w \leq \mathfrak{H}_a \inf_{B_{ar}} w,$$

for any ball  $B_r \subset \Omega_\lambda$  (see [GT, Theorem 8.20]); the Harnack constant  $\mathfrak{H}_a$  can be bounded by the power  $\sqrt{N} + \sqrt{r\mathfrak{L}}$  of a constant only depending on  $N$  and  $a$  (see [GT]). For instance, if  $c(x) \equiv 0$ , by the explicit Poisson's representation formula for harmonic functions, we have that

$$\sup_{B_{ar}} w \leq \left( \frac{1+a}{1-a} \right)^N \inf_{B_{ar}} w,$$

for any  $B_r \subset \Omega_\lambda$  (see [GT] and [DBGV]).

Now, we review the quantitative study of the method of moving planes established in [CMS2], partly based on the work in [ABR]. As already mentioned in the Introduction, the stability of the radial configuration for problem (1.1), (1.3) is obtained in (1.7) in terms of the Lipschitz seminorm on parallel surfaces to  $\Gamma$ .

For a fixed direction  $\omega$  we consider the critical positions and the corresponding points  $p$  and  $q$ , as detailed in (S1) and (S2). As shown in [MS3], the method of moving planes can be applied to  $G$  instead of  $\Omega$ , and the tangency points of cases (S1) and (S2) are  $P$  and  $Q$ , respectively. It is clear that if an estimate like (1.7) holds for  $G$ , then the same holds for  $\Omega$ , since the difference of the radii does not change.

The procedure to obtain (1.7) is quite delicate. For the reader's convenience, we give an outline of it, in which we identify 8 salient steps.

- (i) Following the proof of [CMS2, Theorem 3.3], we show that the values of  $w(P)/\text{dist}(P, \pi_\lambda)$ , in case (S1), and of the partial derivative  $w_\omega(Q)$ , in case (S2), are bounded by some constant times  $[u]_{\partial G}$ .
- (ii) By Harnack's inequality, the smallness obtained in (i) at the points  $P$  and  $Q$  propagates to any point in  $G_\lambda$  sufficiently far from  $\partial G_\lambda$  (see [CMS2, Lemma 3.1]).
- (iii) By using Carleson's inequality, the estimation obtained in the previous step extends to any point in the cap  $G_\lambda$  (see [CMS2, Lemma 3.1]), thus obtaining the inequality

$$\|w\|_{L^\infty(G_\lambda)} \leq C[u]_{\partial G}.$$

Here, the key remark is that  $C$  only depends on  $N$ ,  $\mathfrak{r}$ , the diameter and the  $C^2$ -regularity of  $G$ , but *does not* depend on the particular direction  $\omega$  chosen.

- (iv) The union of  $G \cap \overline{\mathcal{H}_\lambda}$  with its reflection in  $\pi_\lambda$  defines a set  $X$ , symmetric in the direction  $\omega$ , that approximates  $G$ , since the smallness of  $w$  bounds that of  $u$  on  $\partial X$ .
- (v) Since  $u$  is the solution of (1.1), then  $u(x)$  grows linearly with  $d_\Omega(x)$ , when  $x$  moves inside  $\Omega$  from  $\Gamma$ ; this implies that  $u$  can not be too small on  $\partial G = \Gamma^\mathfrak{r}$ .
- (vi) By using both steps (iv) and (v), we find that the distance of every point in  $\partial X$  from  $\partial G$  is not greater than some constant times  $[u]_{\partial G}$  ([CMS2, Lemma 3.4]). This means that  $X$  *fits well*  $G$ , in the sense

that  $X$  contains the parallel set  $G^\sigma$  (related to  $G$ ) for some positive (small) number  $\sigma$  controlled by  $[u]_{\partial G}$  ([CMS2, Theorem 3.5]). This fact is what we call a *quantitative approximate symmetry* of  $G$  in the direction  $\omega$ .

- (vii) An approximate center of symmetry  $\mathfrak{o}$  is then determined as the intersection of  $N$  mutually orthogonal critical hyperplanes. As shown in [ABR, Proposition 6], the distance between  $\mathfrak{o}$  and any other critical hyperplane can be uniformly bounded in terms of the parameter  $\sigma$  in item (vi) and hence of  $[u]_{\partial G}$ .
- (viii) The point  $\mathfrak{o}$  is finally chosen as the center of the spherical annulus  $\{x \in \mathbb{R}^N : r_i < |x - \mathfrak{o}| < r_e\}$  and the estimate (1.7) follows from [ABR, Proposition 7].

Based on this plan, to improve (1.7), it is sufficient to work on the estimates in step (ii). This will be done in Section 3, by refining our use of Harnack's and Carleson's inequalities. As a matter of fact, in [CMS2] we merely used a standard application of Harnack's inequality, by constructing a Harnack's chain of balls of suitably chosen fixed radius  $\mathfrak{r}$ . This strategy only yields an exponential dependence on  $\mathfrak{r}^{-1}$  of the constant in (1.7). In [CMV], we improved these estimates by choosing a chain of balls with radii that decay linearly when the balls approach  $\Gamma$ ; however, this could be done only when  $\Omega$  is convex (or little more) and leads to a Hölder type dependency on  $\mathfrak{r}^{-1}$  of the constant in (1.7).

In Section 3 instead, we use the following plan, that we sketch for the case  $\omega = e_1$  and  $\lambda = 0$ ;  $p$  and  $q$  are the points defined in (S1) and (S2).

We fix  $\mathfrak{r} = \mathfrak{r}_\Omega/4$ , so that  $G = \Omega^{\mathfrak{r}\omega/4}$ . For any  $0 < \delta < \mathfrak{r}_\Omega/4$ , the values of  $w(x)/x_1$  at the points  $p^\delta$  and  $P = p^\mathfrak{r}$  can be compared in the following way

$$w(P)/P_1 \leq C \delta^{-1} w(p^\delta)/p_1^\delta,$$

where  $C$  is a constant not depending on  $\delta$ . Correspondingly, we prove that

$$w_{x_1}(Q) \leq C_G \delta^{-1} w_{x_1}(q^\delta),$$

where  $Q = q^\mathfrak{r}$ . Then, by exploiting steps (i) and (iii), we obtain that

$$(2.6) \quad w \leq C_G \delta^{-1} [u]_{\Gamma^\delta} \quad \text{on the maximal cap of } G_\lambda,$$

where the constant  $C_G$  is the one obtained in step (iii) by letting  $\mathfrak{r} = \mathfrak{r}_\Omega/4$ , and hence it does not depend on  $\delta$ .

Once this work is done, steps (iv)–(viii) can be repeated to find the improved approximate symmetry for the parallel set  $G$ , which clearly implies that for  $\Omega$ . We underline the fact that the dependence on  $\delta$  in (2.6) is optimal, as [CM] indicates.

### 3. ENHANCED STABILITY ESTIMATES

In this section, we line up the major changes needed to obtain (1.7); they only concern step (ii). The following lemma will be useful in the sequel.

**Lemma 3.1.** *Let  $p$  and  $q$  be the points defined in (S1) and (S2), respectively.*

*If  $B$  is a ball of radius  $\mathfrak{r}_\Omega$ , contained in  $\Omega$  and such that  $p$  or  $q$  belong to  $\partial B$ , then the center of  $B$  must belong to  $\overline{\Omega}_\lambda$ .*

*Proof.* The assertion is trivial for case (S2). If case (S1) occurs, without loss of generality, we can assume that  $\omega = e_1$  and  $\lambda = 0$ . Since (S1) holds, the reflected point  $p'$  lies on  $\partial\Omega$  and cannot fall inside  $B$ , since  $p \in \partial B$  and  $B \subset \Omega$ .

Thus, if  $c$  is the center of  $B$ , we have that  $|c - p'| \geq \mathfrak{r}_\Omega = |c - p|$  and hence  $|c_1 + p_1| \geq |c_1 - p_1|$ , which implies that  $c_1 \geq 0$ , being  $p_1 > 0$ .  $\square$

Our first estimate is a quantitative version of Hopf's lemma, that will be useful to treat both occurrences (S1) and (S2).

**Lemma 3.2.** *Let  $B_R = \{x \in \mathbb{R}^N : |x| < R\}$  and, for  $p \in \partial B_R$  and  $s \in (0, R)$ , set  $p^s = p + s\nu(p) = (1 - s/R)p$ .*

*Let  $c \in L^\infty(B_R)$  and suppose that  $w \in C^0(B_R \cup \{p\}) \cap C^2(B_R)$  satisfies the conditions:*

$$\Delta w + c(x)w = 0 \quad \text{and} \quad w \geq 0 \quad \text{in } B_R.$$

*Then, there is a constant  $A = A(N, R, \|c\|_\infty)^1$  such that*

$$(3.1) \quad w(0) \leq A s^{-1} w(p^s) \quad \text{for any } s \in (0, R/2).$$

*Moreover, if  $w(p) = 0$ , then*

$$(3.2) \quad w(0) \leq A w_\nu(p).$$

*Proof.* We proceed as in the standard proof of Hopf's boundary point lemma.

Notice that  $w$  also satisfies

$$\Delta w - c^-(x)w \leq 0 \quad \text{in } B_R,$$

where  $c^-(x) = \max(-c(x), 0)$ . Thus, the strong maximum principle implies that  $w > 0$  in  $B_R$  (unless  $w \equiv 0$ , in which case the conclusion is trivial).

For a fixed  $a \in (0, 1)$  and some parameter  $\alpha > 0$ , set

$$v(x) = \frac{|x|^{-\alpha} - R^{-\alpha}}{(a^{-\alpha} - 1)R^{-\alpha}} \quad \text{for } aR \leq |x| \leq R;$$

notice that  $v > 0$  in  $B_R$ ,  $v = 0$  on  $\partial B_R$  and  $v = 1$  on  $\partial B_{aR}$ . For  $aR < |x| < R$  we then compute that

$$\begin{aligned} \Delta v - c^-(x)v &\geq \frac{\alpha^2 - (N-2)\alpha - |x|^2 c^-(x)}{(a^{-\alpha} - 1)R^{-\alpha}} |x|^{-\alpha-2} \\ &\geq \frac{\alpha^2 - (N-2)\alpha - R^2 \|c^-\|}{(a^{-\alpha} - 1)R^{-\alpha}} |x|^{-\alpha-2}. \end{aligned}$$

Hence, we see that

$$\Delta v - c^-(x)v \geq 0 \quad \text{in } B_R \setminus \overline{B_{aR}},$$

if we choose

$$\alpha = \frac{N-2 + \sqrt{(N-2)^2 + 4R^2 \|c^-\|}}{2}.$$

With this choice of  $\alpha$ , the function

$$z = w - \left[ \min_{\partial B_{aR}} w \right] v$$

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<sup>1</sup>See the proof for its expression.



satisfies the inequalities:

$$\Delta z - c^-(x)z \leq 0 \quad \text{in } B_R \setminus \overline{B_{aR}} \quad \text{and} \quad z \geq 0 \quad \text{on } \partial(B_R \setminus B_{aR}).$$

Thus, the maximum principle gives that  $z \geq 0$  and hence that

$$\min_{\partial B_{aR}} w \leq \frac{w(x)}{v(x)},$$

for  $x \in B_R \setminus \overline{B_{aR}}$ .

Now, choose  $x = p^s$  (since we want that  $p^s \in B_R \setminus \overline{B_{aR}}$ , the constraint  $s/R < 1 - a$  is needed); we thus have that

$$(3.3) \quad \min_{\partial B_{aR}} w \leq \frac{a^{-\alpha} - 1}{(1 - s/R)^{-\alpha} - 1} w(p^s) \leq R \frac{a^{-\alpha} - 1}{\alpha} \frac{w(p^s)}{s},$$

where the last inequality holds for the convexity of the function  $t \mapsto t^{-\alpha}$ .

Harnack's inequality (2.5) then yields

$$w(0) \leq \sup_{B_{aR}} w \leq \mathfrak{H}_a \inf_{B_{aR}} w \leq R \mathfrak{H}_a \frac{a^{-\alpha} - 1}{\alpha} \frac{w(p^s)}{s}.$$

Consequently, by choosing  $a = 1/2$  and setting  $A = R \mathfrak{H}_{1/2} (2^\alpha - 1)/\alpha$ , we readily obtain (3.1).

Finally, if  $w(p) = 0$ , we readily obtain (3.2) from (3.1) and by letting  $s$  go to zero.  $\square$

The following result is crucial to treat the case (S2).

**Lemma 3.3.** *Set  $B_R^+ = \{x \in B_R : x_1 > 0\}$  and  $T = \{x \in \partial B_R^+ : x_1 = 0\}$ . For any point  $q \in \partial B_R \cap T$  and  $s \in [0, R)$ , define  $q^s = q + s\nu(q) = (1 - s/R)q$ .*

*Let  $c \in L^\infty(B_R)$  and suppose  $w \in C^2(B_R^+) \cap C^1(B_R^+ \cup T)$  satisfies the conditions:*

$$\Delta w + c(x)w = 0 \quad \text{and} \quad w \geq 0 \quad \text{in } B_R^+, \quad w = 0 \quad \text{on } T.$$

*Then, there is a constant  $A^* = A^*(N, R, \|c\|_\infty)$  such that*

$$(3.4) \quad w_{x_1}(0) \leq A^* s^{-1} w_{x_1}(q^s) \quad \text{for any } s \in (0, R/2].$$

*Proof.* As in Lemma 3.2, we can assume that  $w > 0$  in  $B_R^+$ .

Inequality (3.4) will be the result of a chain of estimates: with this goal, we introduce the half-annulus  $\mathcal{A}^+ = B_R^+ \setminus \overline{B_\rho^+}$  and the cube

$$Q_\sigma = \{(x_1, \dots, x_N) \in \mathbb{R}^N : 0 < x_1 < 2\sigma, |x_i| < \sigma, i = 2, \dots, N\}.$$

For the moment, we choose  $0 < \rho < R$  and  $0 < \sigma \leq R/\sqrt{N+3}$ , that is in such a way that  $Q_\sigma \subset B_R^+$ ; the precise value of  $\rho$  will be specified later.

The first estimate of our chain is (3.5) below; in order to prove it, we introduce the auxiliary function

$$v(x) = [|x|^{-\alpha} - R^{-\alpha}] x_1 \quad \text{for } x \in \overline{\mathcal{A}^+}.$$

It is clear that  $v > 0$  in  $\mathcal{A}^+$  and  $v = 0$  on  $\partial B_R^+$ ; also, we can choose  $\alpha > 0$  so that  $\Delta v - c^-(x)v \geq 0$  in  $\mathcal{A}^+$  ( $\alpha = (N + \sqrt{N^2 + 4R^2\|c^-\|_\infty})/2$  will do).

We then consider the function  $w/v$  on  $\partial B_\rho^+$ : it is surely well-defined, positive and continuous in  $\partial B_\rho^+ \setminus T$ ; also, it can be extended to be a continuous function up to  $T \cap \partial B_\rho^+$  by defining it equal to its limiting values

$w_{x_1}(x)/(\rho^{-\alpha} - R^{-\alpha})$  for  $x \in T \cap \partial B_\rho^+$ . With this settings,  $w/v$  also turns out to be positive on the whole  $\partial B_\rho^+$  since, on  $T \cap \partial B_\rho^+$ , it is positive by a standard application of Hopf lemma.

These remarks tell us that the minimum of  $w/v$  on  $\partial B_\rho^+$  is well-defined and positive, and hence that the function

$$z = w - \min_{\partial B_\rho^+}(w/v) v$$

satisfies the inequalities

$$\Delta z - c^-(x) z \leq 0 \quad \text{in } \mathcal{A}^+ \quad \text{and} \quad z \geq 0 \quad \text{on } \partial \mathcal{A}^+.$$

Thus, by the maximum principle,  $z \geq 0$  on  $\overline{\mathcal{A}^+}$  and hence

$$\min_{\partial B_\rho^+}(w/v) \leq w(x)/v(x) \quad \text{for every } x \in \overline{\mathcal{A}^+}.$$

For  $s < R - \rho$ , we then can take  $x = q^s + \varepsilon e_1$ , with  $\varepsilon > 0$  so small that  $x \in \mathcal{A}^+$ , take the limit as  $\varepsilon \rightarrow 0$  and, since  $w(q^s) = v(q^s) = 0$ , obtain the inequality

$$\min_{\partial B_\rho^+}(w/v) \leq \frac{w_{x_1}(q^s)}{v_{x_1}(q^s)} = \frac{w_{x_1}(q^s)}{(R-s)^{-\alpha} - R^{-\alpha}}.$$

Again, by the convexity of  $t \mapsto t^{-\alpha}$ , we find that

$$\min_{\partial B_\rho^+}(w/v) \leq \frac{R^{\alpha+1}}{\alpha s} w_{x_1}(q^s),$$

and hence it holds that

$$(3.5) \quad \min_{x \in \partial B_\rho^+} \frac{w(x)}{x_1} \leq \frac{R^{\alpha+1}(\rho^{-\alpha} - R^{-\alpha})}{\alpha s} w_{x_1}(q^s).$$

The second estimate (3.6) below shows that, up to a constant, the minimum in (3.5) can be bounded from below by the value of  $w$  at the center of the cube  $Q_\sigma$ . To do this, we let  $y \in \partial B_\rho^+$  be a point at which the minimum in (3.5) is attained and set

$$\hat{y} = (0, y_2, \dots, y_N) \quad \text{and} \quad \bar{y} = (\sigma, y_2, \dots, y_N);$$

notice that  $\hat{y}$  and  $y$  coincide when  $y_1 = 0$ .

The ball  $B_\sigma(\bar{y})$  is tangent to  $\partial B_R^+ \cap T$  at  $\hat{y}$  and we can choose  $\rho$  such that  $B_\sigma(\bar{y}) \subset B_R^+$ ; thus, by applying Lemma 3.2 to  $B_\sigma(\bar{y})$  with  $p = \hat{y}$ ,  $p^s = y$  and  $\nu = e_1$ , we obtain that

$$w(\bar{y}) \leq A w(y)/y_1 \quad \text{if } y_1 > 0,$$

and

$$w(\bar{y}) \leq A w_{x_1}(y) \quad \text{if } y_1 = 0, \quad \text{being } w(\hat{y}) = 0.$$

Thus, we have proved that

$$w(\bar{y}) \leq A \min_{x \in \partial B_\rho^+} \frac{w(x)}{x_1}.$$

Moreover, if we also choose  $\rho$  such that  $B_\rho(\bar{y}) \subset B_{2\rho}(\bar{y}) \subset B_R^+$ , since the point  $\sigma e_1 \in B_\rho(\bar{y})$ , Harnack's inequality shows that

$$w(\sigma e_1) \leq \mathfrak{H}_{1/2} w(\bar{y}),$$

and hence we obtain that

$$(3.6) \quad w(\sigma e_1) \leq A \mathfrak{H}_{1/2} \min_{x \in \partial B_\rho^+} \frac{w(x)}{x_1}.$$

To conclude the proof, we use two estimates contained in [BCN] (see also [Ba]). First, after some rescaling, we can apply [BCN, Lemma 2.1] to the square  $Q_{\sigma/2}$  and obtain that

$$(3.7) \quad w(t\sigma/2 e_1) \leq t C_1 \frac{\max w}{Q_{\sigma/2}},$$

for every  $t \in (0, 1)$ , where  $C_1$  is the constant in [BCN, Lemma 2.1] that, in our case, only depends on  $N$ ,  $\|c\|_\infty$  and  $R$  (by means of  $\sigma$ ). Thus, since  $w(0) = 0$ , taking the limit as  $t \rightarrow 0^+$  gives that

$$(3.8) \quad w_{x_1}(0) \leq \frac{2C_1}{\sigma} \max w.$$

Secondly, we consider the cube  $Q_\sigma$  and again after some rescaling, we use the Carleson-type estimate [BCN, Theorem 1.3] to obtain that

$$(3.9) \quad \max_{Q_{\sigma/2}} w \leq 2^q B w(\sigma e_1),$$

where, in our case, the constants  $B$  and  $q$  in [BCN, Eq.(1.6)] again only depend on  $N$ ,  $R$  and  $\|c\|_\infty$ . Thus, by (3.8) we have that

$$(3.10) \quad w_{x_1}(0) \leq \frac{2^{q+1} B C_1}{\sigma} w(\sigma e_1).$$

Therefore, by applying (3.10), (3.6) and (3.5), inequality (3.4) holds with

$$A^* = 2^{q+1} A \mathfrak{H}_{1/2} B C_1 \frac{(R/\rho)^\alpha - 1}{\alpha} \frac{R}{\sigma},$$

where the constants  $\rho$  and  $\sigma$  can be chosen as specified along the proof.  $\square$

For the treatment of case (S1), we must pay attention to the fact that the point of tangency  $p$  may be very close to  $\pi_\lambda$  and the interior touching ball at  $p$  may not be contained in the cap. For this reason, we need the following lemma which gives a uniform treatment of all cases occurring when (S1) takes place.

**Lemma 3.4.** *Let  $\xi = \xi_1 e_1$  with  $\xi_1 > 0$  and set*

$$B_R^+(\xi) = \{x \in \mathbb{R}^N : |x - \xi| < R, x_1 > 0\}, \quad T = \{x \in \mathbb{R}^N : |x - \xi| < R, x_1 = 0\}.$$

*For  $p \in \partial B_R^+(\xi) \setminus T$ , define  $p^s$  as in Lemma 3.2.*

*Let  $c$  be essentially bounded on  $B_R(\xi)$  and suppose that  $w \in C^2(B_R^+(\xi)) \cap C^0(B_R^+(\xi) \cup T)$  satisfies*

$$\Delta w + c(x) w = 0 \quad \text{and} \quad w \geq 0 \text{ in } B_R^+(\xi), \quad w = 0 \text{ on } T.$$

*Then, there is a constant  $A^\# = A^\#(N, R, \|c\|_\infty)$  such that*

$$(3.11) \quad w(\xi)/\xi_1 \leq A^\# s^{-1} w(p^s)/p_1^s \quad \text{for any } s \in (0, R/2].$$

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<sup>2</sup>Notice that  $T$  may be the empty set.

*Proof.* We proceed similarly to the proof of Lemma 3.3, with some modifications. We shall still use the cube  $Q_\sigma$ , but we will instead consider the half annulus  $\mathcal{A}^+ = B_R^+(\xi) \setminus \overline{B_\rho^+}(\xi)$ .

Next, we change the auxiliary function  $v$ :

$$v(x) = [|x - \xi|^{-\alpha} - R^{-\alpha}] x_1;$$

of course  $v = 0$  on  $\partial B_R^+(\xi)$  and we can still choose  $\alpha$  so large that  $v$  satisfies the inequality  $\Delta v - c^-(x)v \geq 0$  in  $\overline{\mathcal{A}^+}$ . Thus, the function<sup>3</sup>

$$z = w - \min_{\partial B_\rho^+} (w/v) v$$

is such that  $\Delta z - c^-(x)z \leq 0$  in  $\mathcal{A}^+$  and  $z \geq 0$  on  $\partial \mathcal{A}^+$ . By the maximum principle, we obtain that  $z \geq 0$  on  $\overline{\mathcal{A}^+}$  and hence

$$\min_{\partial B_\rho^+(\xi)} (w/v) \leq w(x)/v(x) \text{ for every } x \in \overline{\mathcal{A}^+}.$$

Again, by arguing as in the proof of Lemma 3.3, we find that

$$(3.12) \quad \min_{x \in \partial B_\rho^+(\xi)} \frac{w(x)}{x_1} \leq \frac{R^{\alpha+1}(\rho^{-\alpha} - R^{-\alpha})}{\alpha s} \frac{w(p^s)}{p_1^s}.$$

Now, to conclude the proof we will treat the cases  $\xi_1 \geq 2\rho$  and  $\xi_1 \leq 2\rho$ , separately.

If  $\xi_1 \geq 2\rho$ , the ball  $B_{2\rho}(\xi)$  is contained in  $B_R^+(\xi)$  and hence, by Harnack's inequality, we have:

$$\frac{w(\xi)}{\xi_1} \leq \frac{w(\xi)}{2\rho} \leq \frac{\mathfrak{H}_{1/2}}{2\rho} \min_{\partial B_\rho^+(\xi)} w \leq \frac{\eta + \rho}{2\rho} \mathfrak{H}_{1/2} \min_{x \in \partial B_\rho^+(\xi)} \frac{w(x)}{x_1}.$$

Thus, (3.12) gives that

$$\frac{w(\xi)}{\xi_1} \leq (\eta + \rho) \mathfrak{H}_{1/2} \frac{R}{\rho} \frac{(R/\rho)^\alpha - 1}{4\alpha s} \frac{w(p^s)}{p_1^s}.$$

If  $\xi_1 \leq 2\rho$ , we repeat the arguments of the last part of the proof of Lemma 3.3, with some slight modification. We take a point  $y \in \partial B_\rho^+(\xi)$  at which the minimum in (3.12) is attained and set  $\bar{y} = (\sigma, y_2, \dots, y_N)$ ,  $\hat{y} = (0, y_2, \dots, y_N)$ . We apply Lemma 3.2 to  $B_\sigma(\bar{y})$ , with  $p = \hat{y}$ ,  $p^s = y$  and  $\nu = e_1$  and, by inspecting the two cases  $y_1 > 0$  and  $y_1 = 0$ , we obtain that

$$w(\bar{y}) \leq A \min_{x \in \partial B_\rho^+(\xi)} \frac{w(x)}{x_1}.$$

As before, we choose  $\rho$  such that  $B_\rho(\bar{y}) \subset B_{2\rho}(\bar{y}) \subset B_R^+$  and, since  $\sigma e_1 \in B_\rho(\bar{y})$ , by Harnack's inequality we find that  $w(\sigma e_1) \leq \mathfrak{H}_{1/2} w(\bar{y})$ , and hence

$$(3.13) \quad w(\sigma e_1) \leq \mathfrak{H}_{1/2} A \min_{x \in \partial B_\rho^+(\xi)} \frac{w(x)}{x_1}.$$

Now, we apply [BCN, Theorem 1.3] to the cube  $Q_\sigma$  and obtain (3.9) as before. Moreover, again we use [BCN, Lemma 2.1] in the cube  $Q_{\sigma/2}$ ; if  $\rho$

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<sup>3</sup>As in the proof of Lemma 3.3, we observe that the function  $w/v$  can be extended continuously on  $T$  and  $w/v > 0$  on  $\partial B_\rho^+(\xi)$ .

is sufficiently small, we have that  $\xi_1 \leq 2\rho \leq \sigma/2$  and hence, applying (3.7) with  $t = 2\xi_1/\sigma$  gives that

$$\frac{w(\xi)}{\xi_1} \leq C_1 \max_{\overline{Q}_{\sigma/2}} w;$$

therefore, (3.9) yields:

$$\frac{w(\xi)}{\xi_1} \leq 2^q B C_1 w(\sigma e_1).$$

From (3.13) and (3.12), we conclude in this case, as well. The constant  $A^\#$  can be computed by suitably choosing  $\rho$  and  $\sigma$  according to the instructions specified in the proof.  $\square$

#### 4. APPROXIMATE SYMMETRY

In this section, we assist the reader to adapt the theorems obtained in [CMS2] in order to prove our new result of approximate symmetry for  $\Omega$ . First, we prove the analogue of [CMS2, Theorem 3.3], that gives an estimate on the symmetry of  $\Omega$  in a fixed direction.

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with boundary  $\Gamma$  of class  $C^{2,\tau}$ ,  $0 < \tau < 1$ , and set  $G = \Omega^{\mathfrak{r}_\Omega/4}$ . For a unit vector  $\omega \in \mathbb{R}^N$ , let  $G_\lambda$  and  $\Omega_\lambda$  be the maximal caps in the direction  $\omega$  for  $G$  and  $\Omega$ , respectively.*

*Let  $u \in C^{2,\tau}(\overline{\Omega})$  be a solution of (1.1) and let  $w$  be defined by (2.4).*

*Then, for every  $\delta \in (0, \mathfrak{r}_\Omega/8)$ , we have that*

$$(4.1) \quad w^\lambda \leq C\delta^{-1}[u]_{\Gamma^\delta} \quad \text{on (a connected component of) } G_\lambda.$$

*Here,  $C$  is a constant depending on  $N$ ,  $\mathfrak{L}$ ,  $\mathfrak{d}_\Omega$  and the  $C^{2,\tau}$ -regularity of  $\Gamma$ .*

*Proof.* We point out that  $G$  is connected. Also, as already done before, we can assume that  $\omega = e_1$  and  $\lambda = 0$ .

Let  $p$  and  $q$  be the points defined in (S1) and (S2), respectively;  $P$  and  $Q$  are the points in  $\partial G$  already defined.

In what follows, we chose to still denote by  $\Omega_\lambda$  and  $G_\lambda$  the connected components of the maximal caps  $\Omega_\lambda$  and  $G_\lambda$  that intersect  $B_{\mathfrak{r}_\Omega/4}(P)$ , if case (S1) occurs, and the connected components of  $\Omega_\lambda$  and  $G_\lambda$  that intersect  $B_{\mathfrak{r}_\Omega/4}(Q)$ , if case (S2) occurs.

Lemma 3.1 ensures that the interior ball of radius  $\mathfrak{r}_\Omega$  touching  $\partial\Omega$  at  $p$  or  $q$  has its center in  $\overline{\Omega}_\lambda$ ; hence,  $P \in \Omega_\lambda$  and  $Q \in \partial\Omega_\lambda \cap \pi_\lambda$ . We then apply [CMS2, Lemma 4.2] with the following settings:  $D_1 = G_\lambda$ ,  $D_2 = \Omega_\lambda$ ,  $R = \mathfrak{r}_\Omega/4$ , and  $z = P$ , if case (S1) occurs, and  $z = Q$ , if case (S2) occurs. Thus, we find that

$$(4.2) \quad w(x) \leq C w(P)/P_1 \quad \text{for } x \in \overline{G}_\lambda,$$

and

$$(4.3) \quad w(x) \leq C w_{x_1}(Q) \quad \text{for } x \in \overline{G}_\lambda,$$

respectively. Here, the constant  $C$  depends only on  $N$ ,  $\mathfrak{r}_\Omega$ ,  $\mathfrak{L}$  and  $\mathfrak{d}_\Omega$ .

If (S1) occurs, we apply Lemma 3.4 by letting  $R = \mathfrak{r}_\Omega/4$  and  $\xi = P$  (this is always possible after a translation in a direction orthogonal to  $e_1$ ), and from (4.2) we obtain that

$$(4.4) \quad w(x) \leq C A^\# \delta^{-1} w(p^\delta)/p_1^\delta \quad \text{for } x \in \overline{G}_\lambda,$$

for any  $\delta \in (0, \mathfrak{r}_\Omega/8)$ .

If (S2) occurs, we apply instead Lemma 3.3 (with  $\xi = 0$  and  $R = \mathfrak{r}_\Omega/4$ ) and (4.3): we find that

$$(4.5) \quad w(x) \leq C A^* \delta^{-1} w_{x_1}(q^\delta) \quad \text{for } x \in \overline{G}_\lambda,$$

for any  $\delta \in (0, \mathfrak{r}_\Omega/4)$ .

The rest of the proof runs similarly to that of [CMS2, Theorem 3.3], where the estimates of [CMS2, Lemma 3.2] should be replaced by (4.4) and (4.5). For the reader's convenience, we give a sketch of the proof with the usual settings ( $\omega = e_1$  and  $\lambda = 0$ ). In particular, we show how to relate  $w(p^\delta)/p_1^\delta$  and  $w_{x_1}(q^\delta)$  to  $[u]_{\Gamma^\delta}$ , which is the main argument of the proof.

Let us assume that case (S1) occurs. If  $p_1^\delta \geq \mathfrak{r}_\Omega/2$ , since  $p^\delta$  and its reflection  $(p^\delta)'$  about  $\pi_\lambda$  lie on  $\Gamma^\delta$ , then

$$w(p^\delta) = u((p^\delta)') - u(p^\delta) \leq \mathfrak{d}_\Omega [u]_{\Gamma^\delta},$$

and hence we easily obtain that

$$(4.6) \quad w(p^\delta)/p_1^\delta \leq 2\mathfrak{d}_\Omega \mathfrak{r}_\Omega^{-1} [u]_{\Gamma^\delta}.$$

If  $p_1^\delta < \mathfrak{r}_\Omega/2$ , then  $|p^\delta - (p^\delta)'| < \mathfrak{r}_\Omega$ , then every point of the segment joining  $(p^\delta)'$  to  $p^\delta$  is at a distance not greater than  $\mathfrak{r}_\Omega$  from some connected component of  $\Gamma^\delta$ . The curve  $\gamma$  obtained by projecting that segment on that component has length bounded by  $\widehat{C} |p^\delta - (p^\delta)'|$ , where  $\widehat{C}$  is a constant depending on  $\mathfrak{r}_\Omega$  and the regularity of  $\Gamma^\delta$  (and hence on the regularity of  $\Gamma$  since  $\delta < \mathfrak{r}_\Omega/8$ ). An application of the mean value theorem to the restriction of  $u$  to  $\gamma$  gives that  $u((p^\delta)') - u(p^\delta)$  can be estimated by the length of  $\gamma$  times the maximum of the tangential gradient of  $u$  on  $\Gamma^\delta$ . Thus,

$$(4.7) \quad w(p^\delta) \leq 2\widehat{C} p_1^\delta [u]_{\Gamma^\delta}.$$

Therefore, (4.6) and (4.7) yield the conclusion, if case (S1) is in force.

Case (S2) is simpler. Since  $e_1$  belongs to the tangent hyperplane to  $\Gamma_\delta$  at  $q^\delta$ , we readily obtain (4.1).  $\square$

As outlined in Section 2, Theorem 4.1 completes steps (i)-(iii) and leads to stability bounds for the symmetry in one direction. Now, we complete steps (iv)-(viii).

As described in steps (iv) and (vi), we define a symmetric open set  $X$  and show that  $G$  is almost equal to  $X$ . In order to do that, we need a priori bounds on  $u$  from below in terms of the distance function from  $\partial G$ , as specified in (v). As observed in [ABR] and [CMS2], such a bound requires a positive lower bound for  $u_\nu$  on  $\Gamma$ ,

$$u_\nu \geq \mathfrak{c}_0 \quad \text{on } \Gamma.$$

If  $f(0) > 0$ , this is guaranteed by Hopf lemma. If  $f(0) \leq 0$  instead, such a bound must be introduced as an assumption, as it can be realized by considering any (positive) multiple of the first Dirichlet eigenfunction  $\phi_1$  for

$-\Delta$ . In fact, for any  $n \in \mathbb{N}$  the function  $\phi_1/n$  satisfies (1.1) with  $f(u) = \lambda_1 u$ , being  $\lambda_1$  the first Dirichlet eigenvalue, and it is clear that, although  $(\phi_1/n)_\nu \rightarrow 0$  on  $\Gamma$  as  $n \rightarrow \infty$ , one cannot expect to derive any information on the shape of  $\Omega$ .

The final stability result, step (viii), is obtained by defining an approximate center of symmetry  $\mathfrak{o}$  as the intersection of  $N$  orthogonal hyperplanes as described in step (vii) (see also [CMS2, Proof of Theorem 1.1]).

We can now conclude this section with our improved stability estimate on the symmetry of  $\Omega$ .

**Theorem 4.2.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with boundary  $\Gamma$  of class  $C^{2,\tau}$ ,  $0 < \tau < 1$ . Let  $u \in C^{2,\tau}(\overline{\Omega})$  be a solution of (1.1).*

*There exist constants  $\varepsilon, C > 0$  and  $\delta_0 \in (0, \mathfrak{r}_\Omega/4)$  such that, if*

$$(4.8) \quad [u]_{\Gamma^{\delta_0}} \leq \varepsilon,$$

*then there are two concentric balls  $B_{r_i}$  and  $B_{r_e}$  such that*

$$(4.9) \quad B_{r_i} \subset \Omega \subset B_{r_e}$$

*and*

$$(4.10) \quad r_e - r_i \leq C \delta^{-1} [u]_{\Gamma^\delta},$$

*for any  $\delta \in (0, \delta_0]$ .*

*The constants  $\varepsilon$  and  $C$  only depend on  $N, \mathfrak{r}_\Omega, \mathfrak{d}_\Omega, \mathfrak{L}, \mathfrak{c}_\mathfrak{o}, \max_{\overline{\Omega}} u$  and the  $C^{2,\tau}$ -regularity of  $\Gamma$ .*

*Proof.* Thanks to Theorem 4.1, we can repeat the argument of the proof of [CMS2, Theorem 4.2] in which we replace formula [CMS2, (3.15)] by (4.1). Hence, there exists two concentric balls  $B_{r_i^*}$  and  $B_{r_e^*}$  and two constants  $\varepsilon$  (independent of  $\delta$ ) such that

$$B_{r_i^*} \subset G \subset B_{r_e^*} \quad \text{and} \quad r_e^* - r_i^* \leq C \delta^{-1} [u]_{\Gamma^\delta},$$

if  $[u]_{\Gamma^\delta} \leq \varepsilon$ . Moreover, since  $\varepsilon$  does not depend on  $\delta$  and

$$\lim_{\delta \rightarrow 0^+} [u]_{\Gamma^\delta} = 0,$$

we can find  $\delta_0 \in (0, \mathfrak{r}_\Omega)$  such that  $[u]_{\Gamma^\delta} \leq \varepsilon$  for any  $\delta \in (0, \delta_0)$ . To complete the proof, we observe that (4.9) and (4.10) hold with  $r_i = r_i^* + \mathfrak{r}_\Omega/4$  and  $r_e = r_e^* + \mathfrak{r}_\Omega/4$ .  $\square$

## 5. SERRIN'S PROBLEM

In this section, we give a new proof of Serrin's symmetry result and a corresponding stability estimate for spherical symmetry by using the improved stability inequality for the parallel surface problem (1.1), (1.3), as just proved in Theorem 4.2. We need the following lemma.

**Lemma 5.1.** *Let  $\Omega$  be a bounded domain with boundary  $\Gamma$  of class  $C^2$  and set  $r = \min(\mathfrak{r}_\Omega, \mathfrak{R}_\Omega)$ . Let  $u$  be of class  $C^2$  in a neighborhood of  $\Gamma$  and such that  $u = 0$  on  $\Gamma$ .*

*Then*

$$(5.1) \quad [u]_{\Gamma^\delta} \leq \frac{\delta}{1 - \delta/r} [u_\nu]_\Gamma + \int_0^\delta \frac{(\delta - t)(r - t)}{r - \delta} [u_{\nu\nu}]_{\Gamma^t} dt,$$

for every  $\delta \in [0, r)$ .

In particular, if  $\delta \leq r/2$ , we have that

$$(5.2) \quad [u]_{\Gamma^\delta} \leq 2\delta \left\{ [u_\nu]_\Gamma + \int_0^\delta [u_{\nu\nu}]_{\Gamma^t} dt \right\}.$$

*Proof.* Let  $p_1$  and  $p_2$  be two points on  $\Gamma$ , so that  $p_i^\delta = p_i + \delta \nu(p_i)$ ,  $i = 1, 2$ , are points on  $\Gamma^\delta$ . It is clear that  $|p_1^\delta - p_2^\delta| \geq |p_1 - p_2| - \delta |\nu(p_1) - \nu(p_2)|$  and hence:

$$(5.3) \quad |p_1^\delta - p_2^\delta| \geq (1 - \delta/r) |p_1 - p_2|.$$

By applying Taylor's formula to the values of  $u$  at  $p_1^\delta$  and  $p_2^\delta$  and taking the difference, we have that

$$u(p_1^\delta) - u(p_2^\delta) = \delta [u_\nu(p_1) - u_\nu(p_2)] + \int_0^\delta (\delta - t) [u_{\nu\nu}(p_1^t) - u_{\nu\nu}(p_2^t)] dt,$$

since  $u = 0$  at  $p_1$  and  $p_2$  and being  $\nu(p_i^t) = \nu(p_i)$ ,  $i = 1, 2$ . Dividing both sides by  $|p_1^\delta - p_2^\delta|$  and using (5.3), gives that

$$\begin{aligned} \frac{|u(p_1^\delta) - u(p_2^\delta)|}{|p_1^\delta - p_2^\delta|} &\leq \frac{\delta}{1 - \delta/r} \frac{|u_\nu(p_1) - u_\nu(p_2)|}{|p_1 - p_2|} + \\ &\quad \int_0^\delta \frac{(\delta - t)(r - t)}{r - \delta} \frac{|u_{\nu\nu}(p_1^t) - u_{\nu\nu}(p_2^t)|}{|p_1^t - p_2^t|} dt \leq \\ &\quad \frac{\delta}{1 - \delta/r} [u_\nu]_\Gamma + \int_0^\delta \frac{(\delta - t)(r - t)}{r - \delta} [u_{\nu\nu}]_{\Gamma^t} dt. \end{aligned}$$

Therefore, (5.1) and hence (5.2) follow at once.  $\square$

We are now in position to prove both symmetry and stability for Serrin's problem. Of course, stability implies symmetry, when the normal derivative of  $u$  is exactly constant on  $\Gamma$ . However, we prefer to present the two results separately.

**Theorem 5.2** (Symmetry). *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with boundary  $\Gamma$  of class  $C^{2,\tau}$ ,  $0 < \tau < 1$ . Let  $u \in C^{2,\tau}(\overline{\Omega})$ , satisfy (1.1) and suppose that (1.2) holds with  $\mathfrak{a} > 0$ .*

*Then  $\Omega$  is a ball.*

*Proof.* The assumed regularity of  $u$  implies that  $[u_{\nu\nu}]_{\Gamma^t} = O(t^{\tau-1})$  as  $t \rightarrow 0$  and hence, since (1.2) is in force, (5.2) tells us that (4.8) holds for some  $\delta_0 > 0$ . Thus, Theorem 4.2 can be applied and, by (5.2), we have that

$$B_{r_i} \subset \Omega \subset B_{r_e} \quad \text{and} \quad r_e - r_i \leq 2C \int_0^\delta [u_{\nu\nu}]_{\Gamma^t} dt$$

for any  $\delta \in (0, \delta_0)$ . The behavior of  $[u_{\nu\nu}]_{\Gamma^t}$  as  $t \rightarrow 0$  then implies that the integral at the right-hand side can be made arbitrarily small. Therefore,  $r_e = r_i$ , that implies that  $\Omega$  is a ball.  $\square$

**Theorem 5.3** (Stability). *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with boundary  $\Gamma$  of class  $C^{2,\tau}$ ,  $0 < \tau < 1$ , and let  $u \in C^{2,\tau}(\overline{\Omega})$  be solution of (1.1). Let  $C$  be the constant in (4.10).*



There are two concentric balls  $B_{r_i}$  and  $B_{r_e}$  such that (4.9) holds with

$$(5.4) \quad r_e - r_i \leq 2C [u_\nu]_\Gamma.$$

In particular, if (1.2) is in force with  $\mathfrak{a} > 0$ , then  $\Omega$  is a ball.

*Proof.* The regularity of  $u$  and (5.2) imply that (4.8) holds for some  $\delta_0 > 0$ . Thus, Theorem 4.2 can be applied and, by (5.2), we have that (4.9) holds with

$$r_e - r_i \leq 2C \left\{ [u_\nu]_\Gamma + \int_0^\delta [u_{\nu\nu}]_\Gamma t \, dt \right\},$$

for every  $\delta \in (0, \delta_0)$ ; (5.4) then follows by letting  $\delta$  tend to 0.  $\square$

*Remark 5.4.* We notice that in Theorem 5.3 we are not assuming the smallness of  $[u_\nu]_\Gamma$  to prove (5.4).

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DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI PALERMO, VIA ARCHIRAFI 34, 90123, ITALY.

*E-mail address:* [giulio.ciraolo@unipa.it](mailto:giulio.ciraolo@unipa.it)

*URL:* <http://www.math.unipa.it/~g.ciraolo/>

DIPARTIMENTO DI MATEMATICA ED INFORMATICA “U. DINI”, UNIVERSITÀ DI FIRENZE, VIALE MORGAGNI 67/A, 50134 FIRENZE, ITALY.

*E-mail address:* [magnanin@math.unifi.it](mailto:magnanin@math.unifi.it)

*URL:* <http://web.math.unifi.it/users/magnanin>

DIPARTIMENTO DI MATEMATICA ED INFORMATICA “U. DINI”, UNIVERSITÀ DI FIRENZE, VIALE MORGAGNI 67/A, 50134 FIRENZE, ITALY.

*E-mail address:* [vespri@math.unifi.it](mailto:vespri@math.unifi.it)

*URL:* <http://web.math.unifi.it/users/vespri>